Inverzna matrica

Transponovanu matricu matrice A obilježavamo sa A^T.
Kofaktor A_{ij}, matrice A, elementa a_{ij} je determinanta pomnožena sa $(-1)^{i+j}$ čiji su elementi svi elementi iz matrice A osim one kolone i one vrste u kojoj se nalazi koeficijent a_{ij}.

Npr.

$$A=\begin{bmatrix}3&7&2\\6&8&9\\1&2&4\end{bmatrix},\quad \underbrace{A_{23}=(-1)^{2+3}\begin{vmatrix}3&7\\1&2\end{vmatrix}}_{\text{kofaktor elementa } a_{23}},\quad \overbrace{A_{12}=(-1)^{1+2}\begin{vmatrix}3&2\\1&4\end{vmatrix}}^{\text{kofaktor elementa } a_{12}},\quad \underbrace{A_{31}=(-1)^{3+1}\cdot\begin{vmatrix}7&2\\8&9\end{vmatrix}}_{\text{kofaktor elementa } a_{31}}$$

$$A^T=\begin{bmatrix}3&6&1\\7&8&2\\2&9&4\end{bmatrix}$$

Kofaktor matrica (A_{kof}) kvadratne matrice A je matrica kofaktora A_{ik} elemenata a_{ik} dane matrice.

$$A=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}\qquad A_{kof}=\begin{bmatrix}A_{11}&A_{12}&A_{13}\\A_{21}&A_{22}&A_{23}\\A_{31}&A_{32}&A_{33}\end{bmatrix}$$

Za matricu A kažemo da je regularna ako je $\det A\neq 0$.
Inverznu matricu računamo po formuli

$$A^{-1}=\frac{1}{\det A}\cdot A_{kof}^{T}$$

Neke osobine inverzne matrice:

$$A^{-1}\cdot A=A\cdot A^{-1}=I$$

$$(AB)^{-1}=B^{-1}A^{-1}$$

1. Naći inverznu matricu matrice $A=\begin{bmatrix}1&0&2\\1&1&0\\1&1&1\end{bmatrix}$.

Rj. $A^{-1}=\frac{1}{\det A}\cdot A_{kof}^{T}$

$$\det A=\begin{vmatrix}1&0&2\\1&1&0\\1&1&1\end{vmatrix}\overset{III_R-II_R}{=\!=}\begin{vmatrix}1&0&2\\1&1&0\\0&0&1\end{vmatrix}=1\cdot\begin{vmatrix}1&0\\1&1\end{vmatrix}=1$$

$$A_{11}=(-1)^2\begin{vmatrix}1&0\\1&1\end{vmatrix}=1\qquad A_{13}=(-1)^4\begin{vmatrix}1&1\\1&1\end{vmatrix}=0\qquad A_{22}=(-1)^4\begin{vmatrix}1&2\\1&1\end{vmatrix}=-1$$

$$A_{12}=(-1)^3\begin{vmatrix}1&0\\1&1\end{vmatrix}=-1\qquad A_{21}=(-1)^3\begin{vmatrix}0&2\\1&1\end{vmatrix}=2\qquad A_{23}=(-1)^5\begin{vmatrix}1&0\\1&1\end{vmatrix}=-1$$

$$A_{31}=(-1)^4\begin{vmatrix}0&2\\1&0\end{vmatrix}=-2 \qquad A_{kof}=\begin{bmatrix}1&-1&0\\2&-1&-1\\-2&2&1\end{bmatrix}$$

$$A_{32}=(-1)^5\begin{vmatrix}1&2\\1&0\end{vmatrix}=2$$

$$A_{33}=(-1)^6\begin{vmatrix}1&0\\1&1\end{vmatrix}=1 \qquad A^{-1}=\begin{bmatrix}1&2&-2\\-1&-1&2\\0&-1&1\end{bmatrix}$$

inverzna matrica matrice A

provjera:

$$\begin{bmatrix}1&0&2\\1&1&0\\1&1&1\end{bmatrix}\cdot\begin{bmatrix}1&2&-2\\-1&-1&2\\0&-1&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

2. Naći inverznu matricu matrice $B=\begin{bmatrix}3&2&4\\2&3&4\\1&1&2\end{bmatrix}$.

Rj. $B^{-1}=\dfrac{1}{\det B}B_{kof}^T$, $\det B=\begin{vmatrix}3&2&4\\2&3&4\\1&1&2\end{vmatrix}\begin{matrix}II_k-I_k\\ \hline III_k-I_k\cdot 2\end{matrix}\begin{vmatrix}3&-1&-2\\2&1&0\\1&0&0\end{vmatrix}=\begin{vmatrix}-1&-2\\1&0\end{vmatrix}=2$

$$B_{11}=(-1)^2\begin{vmatrix}3&4\\1&2\end{vmatrix}=2 \qquad B_{21}=(-1)^2\begin{vmatrix}2&4\\1&2\end{vmatrix}=0 \qquad B_{31}=(-1)^4\begin{vmatrix}2&4\\3&4\end{vmatrix}=-4$$

$$B_{12}=(-1)^3\begin{vmatrix}2&4\\1&2\end{vmatrix}=0 \qquad B_{22}=(-1)^4\begin{vmatrix}3&4\\1&2\end{vmatrix}=2 \qquad B_{32}=(-1)^5\begin{vmatrix}3&4\\2&4\end{vmatrix}=-4$$

$$B_{13}=(-1)^4\begin{vmatrix}2&3\\1&1\end{vmatrix}=-1 \qquad B_{23}=(-1)^5\begin{vmatrix}3&2\\1&1\end{vmatrix}=-1 \qquad B_{33}=(-1)^6\begin{vmatrix}3&2\\2&3\end{vmatrix}=5$$

$$B_{kof}^T=\begin{bmatrix}2&0&-4\\0&2&-4\\-1&-1&5\end{bmatrix},\quad B^{-1}=\frac{1}{2}\begin{bmatrix}2&0&-4\\0&2&-4\\-1&-1&5\end{bmatrix}=\begin{bmatrix}1&0&-2\\0&1&-2\\-\frac{1}{2}&-\frac{1}{2}&\frac{5}{2}\end{bmatrix}$$

tražena inverzna matrica

3. Naći inverznu matricu matrice $C=\begin{bmatrix}2&1\\5&4\end{bmatrix}$.

Rj. $C^{-1}=\dfrac{1}{\det C}C_{kof}^T$, $\det C=\begin{vmatrix}2&1\\5&4\end{vmatrix}=3$

$$C_{11}=(-1)^2\cdot 4=4 \qquad C_{21}=(-1)^3\cdot 1=-1$$

$$C_{12}=(-1)^3\cdot 5=-5 \qquad C_{22}=(-1)^4\cdot 2=2$$

$$C_{kof}^T=\begin{bmatrix}4&-1\\-5&2\end{bmatrix} \qquad C^{-1}=\begin{bmatrix}\frac{4}{3}&-\frac{1}{3}\\-\frac{5}{3}&\frac{2}{3}\end{bmatrix}$$

4. Naći inverznu matricu sljedećih matrica:

a) $A=\begin{bmatrix}3&4&4\\1&6&1\\2&3&3\end{bmatrix}$

b) $B=\begin{bmatrix}-3&-1&-1\\1&3&2\\-2&-1&-2\end{bmatrix}$

c) $C=\begin{bmatrix}7&3&3\\6&3&4\\-1&-2&-3\end{bmatrix}$

Rješenja:

a) $A^{-1}=\begin{bmatrix}3&0&-4\\-\frac{1}{5}&\frac{1}{5}&\frac{1}{5}\\-\frac{8}{5}&-\frac{1}{5}&\frac{14}{5}\end{bmatrix}$

b) $B^{-1}=\begin{bmatrix}-\frac{4}{9}&-\frac{1}{9}&\frac{1}{9}\\-\frac{2}{9}&\frac{4}{9}&\frac{5}{9}\\\frac{5}{9}&-\frac{1}{9}&-\frac{8}{9}\end{bmatrix}$

c) $\det C=8$

Pronalaženje inverzne matrice uz pomoć Gauss-Jordan-ovih eliminacija

Posmatrajmo neku proizvoljnu matricu A.
Gauss-Jordan-ove operacije definisane na proizvoljnoj matrici su
(i) množenje proizvoljnog reda matrice brojem različitim od 0
(ii) dodavanje reda i matrice, pomnožen nekim brojem, redu j $(i \neq j)$

Ako je B matrica dobijena iz A pomoću Gauss-Jordanovih operacija pišemo

$$A \xrightarrow{\text{Gauss-Jordan}} B$$

Vrijedi sljedeća teorema

Teorem (računanje inverza)
Za inverznu matricu matrice A vrijedi sljedeća redukcija

$$[A \mid I] \xrightarrow{\text{Gauss-Jordan}} [I \mid A^{-1}] .$$

Ova redukcija neće raditi jedino u slučaju ako se pojavi red nula na lijevoj strani u matrici A, a ovo će se pojaviti ako i samo ako je A singularna matrica. Drugačiji (i nekako mnogo praktičniji) algoritam za pronalaženje inverzne matrice je pomoću LU faktorizacije.

\# Uz pomoć Gauss-Jordanovih eliminacija izračunati inverznu matricu matrice $Q = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

Rj:

$$[Q \mid I] = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{II_v + I_v \cdot (-1)} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \xrightarrow{I_v + II_v \cdot (-1)}$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

Prema tome $Q^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

\# Uz pomoć Gauss-Jordanovih eliminacija izračunati inverznu matricu matrice $Q = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Rj:

$$[Q \mid I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{I_v + II_v \cdot (-1)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{II_v + III_v \cdot (-1)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow Q^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

(#) Uz pomoć Gauss-Jordanovih eliminacija izračunati inverznu matricu matrice $P=\begin{bmatrix}1&1&1\\1&2&2\\1&2&3\end{bmatrix}$.

Rj.

$$[P\,|\,I]=\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\1&2&2&0&1&0\\1&2&3&0&0&1\end{array}\right]\xrightarrow[III_v+I_v\cdot(-1)]{II_v+I_v\cdot(-1)}\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&1&-1&1&0\\0&1&2&-1&0&1\end{array}\right]$$

$$\xrightarrow{III_v+II_v\cdot(-1)}\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&1&-1&1&0\\0&0&1&0&-1&1\end{array}\right]\xrightarrow{I_v+II_v\cdot(-1)}\left[\begin{array}{ccc|ccc}1&0&0&2&-1&0\\0&1&1&-1&1&0\\0&0&1&0&-1&1\end{array}\right]$$

$$\xrightarrow{II_v+III_v\cdot(-1)}\left[\begin{array}{ccc|ccc}1&0&0&2&-1&0\\0&1&0&-1&2&-1\\0&0&1&0&-1&1\end{array}\right]$$

Prema tome $P^{-1}=\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&1\end{bmatrix}$.

Matrične jednačine

U sljedećim primjerima neka su A, B, C, X neke date kvadratne matrice.

$$A^{-1}\cdot B \neq B\cdot A^{-1}$$

$$A\cdot B \neq B\cdot A$$

Da bismo odredili nepoznatu X u matričnoj jednačini prvo trebamo izvesti formulu za nepoznatu X.

Matrice se ne mogu dijeliti.

\# $A\cdot X = B \quad / \cdot A^{-1}$ sa lijeve strane

$$A^{-1}\cdot A\cdot X = A^{-1}\cdot B$$

$$I\cdot X = A^{-1}\cdot B$$

$$X = A^{-1}\cdot B$$

\# $A\cdot X\cdot B = C \quad / \cdot A^{-1}$ sa lijeve strane

$$A^{-1}\cdot A\cdot X\cdot B = A^{-1}\cdot C$$

$I\cdot X\cdot B = A^{-1}\cdot C \quad / B^{-1}$ sa desne strane

$$X\cdot B\cdot B^{-1} = A^{-1}\cdot C\cdot B^{-1}$$

$$X\cdot I = A^{-1}\cdot C\cdot B^{-1}$$

$$X = A^{-1}\cdot C\cdot B^{-1}$$

\# $A\cdot X + I = X - 2I$

$$AX - X = -I - 2I$$

$$\underbrace{(A-I)}_{B}X = -3I$$

$BX = -3I \quad / \cdot B^{-1}$ sa desne strane

$$B^{-1}\cdot B\cdot X = B^{-1}\cdot(-3I)$$

$$I\cdot X = -3B^{-1}$$

$$X = -3(A-I)^{-1}$$

\# $X^{-1}\cdot A = B^{-1} \quad / \cdot A^{-1}$ sa desne strane

$$X^{-1}\cdot A\cdot A^{-1} = B^{-1}\cdot A^{-1}$$

$$X^{-1}\cdot I = B^{-1}\cdot A^{-1}$$

$X^{-1} = B^{-1}\cdot A^{-1} \quad /^{(-1)}$

$$X = A\cdot B$$

\# $A^{-1}\cdot X = X - I$

$$A^{-1}\cdot X - X = -I$$

$$\underbrace{(A^{-1} - I)}_{B}\cdot X = -I$$

$BX = -I \quad / \cdot B^{-1}$ sa lijeve strane

$$B^{-1}\cdot B\cdot X = -B^{-1}\cdot I$$

$$X = -B^{-1}$$

$$X = -(A^{-1} - I)^{-1}$$

$\underbrace{(A+3I)}_{C}(X-I)=B$

$$C(X-I)=B \quad / \cdot C^{-1} \text{ sa lijeve strane}$$

$$C^{-1}C(X-I)=C^{-1}\cdot B$$

$$X-I=C^{-1}\cdot B$$

$$X=C^{-1}\cdot B+I$$

$$X=(A+3I)^{-1}\cdot B+I$$

$B^{-1}\cdot X\cdot A=(3B+2I)^{-1}$

$$/ \cdot B \text{ sa lijeve strane}$$

$$B\cdot B^{-1}\cdot X\cdot A=B(3B+2\cdot I)^{-1}$$

$$X\cdot A=B(3B+2I)^{-1} \quad / \cdot A^{-1} \text{ sa desne strane}$$

$$X=B(3B+2I)^{-1}\cdot A^{-1}$$

$(AXB)^{-1}=B^{-1}(X^{-1}+B)$

$$/ \cdot (AXB) \text{ sa lijeve strane}$$

$$(AXB)(AXB)^{-1}=AX\underbrace{BB^{-1}}_{I}(X^{-1}+B)$$

$$I=AX(X^{-1}+B)$$

$$I=AXX^{-1}+AXB$$

$$I=A+AXB$$

$$AXB=I-A \quad \begin{array}{l}/ \cdot A^{-1} \text{ sa lijeve str.}\\ \cdot B^{-1} \text{ sa desne str.}\end{array}$$

$$A^{-1}AXB\cdot B^{-1}=A^{-1}(I-A)\cdot B^{-1}$$

$$X=A^{-1}(I-A)\cdot B^{-1}$$

1. Riješiti matričnu jednačinu $X\cdot\begin{bmatrix}1&1&1\\-1&-2&-3\\2&3&5\end{bmatrix}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&-1\end{bmatrix}$.

Rj.

$$X\cdot A=B \quad / \cdot A^{-1} \text{ sa desne str.}$$

$$X=B\cdot A^{-1}, \quad A=\begin{bmatrix}1&1&1\\-1&-2&-3\\2&3&5\end{bmatrix}$$

$$A^{-1}=\frac{1}{\det A}\cdot A_{kof}^{T}$$

$$A_{11}=(-1)^2\begin{vmatrix}-2&-3\\3&5\end{vmatrix}=-1$$

$$A_{12}=(-1)^3\begin{vmatrix}-1&-3\\2&5\end{vmatrix}=-1$$

$$A_{13}=(-1)^4\begin{vmatrix}-1&-2\\2&3\end{vmatrix}=1$$

$$\det A=\begin{vmatrix}1&1&1\\-1&-2&-3\\2&3&5\end{vmatrix}\overset{I_k-III_k}{\underset{II_k-III_k}{=}}\begin{vmatrix}0&0&1\\2&1&-3\\-3&-2&5\end{vmatrix}=\begin{vmatrix}2&1\\-3&-2\end{vmatrix}=-1.$$

$$A_{21}=(-1)^3\begin{vmatrix}1&1\\3&5\end{vmatrix}=-2$$

$$A_{22}=(-1)^4\begin{vmatrix}1&1\\2&5\end{vmatrix}=3$$

$$A_{23}=(-1)^5\begin{vmatrix}1&1\\2&3\end{vmatrix}=-1$$

$$A_{31}=(-1)^4\begin{vmatrix}1&1\\-2&-3\end{vmatrix}=-1$$

$$A_{32}=(-1)^5\begin{vmatrix}1&1\\-1&-3\end{vmatrix}=2$$

$$A_{33}=(-1)^6\begin{vmatrix}1&1\\-1&-2\end{vmatrix}=-1$$

$$A_{kof}^{T}=\begin{bmatrix}-1&-2&-1\\-1&3&2\\1&-1&-1\end{bmatrix} \qquad A^{-1}=\begin{bmatrix}1&2&1\\1&-3&-2\\-1&1&1\end{bmatrix}$$

$$X = B\cdot A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & -3 & -2 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & -2 \\ 1 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$ rješenje matrične jednačine

2. Riješiti matričnu jednačinu $A\cdot X = X + I$ ako je $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 3 & -2 \\ 3 & 1 & -1 \end{bmatrix}$.

Rj.

$AX = X + I$

$AX - X = I$

$(A-I)X = I \quad /\cdot(A-I)^{-1}$ sa lijeve strane

$(A-I)(A-I)^{-1}X = (A-I)^{-1}\cdot I$

$X = (A-I)^{-1}$

$$C = A - I = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 3 & -2 \\ 3 & 1 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & -2 \\ 3 & 1 & -2 \end{bmatrix}$$

$$C^{-1} = \frac{1}{\det C} C_{kof}^{T}$$

$$\det C = \begin{vmatrix} 0 & -1 & 1 \\ 1 & 2 & -2 \\ 3 & 1 & -2 \end{vmatrix} \overset{II_k + III_k}{=\!=} \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & -2 \\ 3 & -1 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = -1$$

$C_{11} = (-1)^2 \begin{vmatrix} 2 & -2 \\ 1 & -2 \end{vmatrix} = -2$ $\quad C_{21} = (-1)^3 \begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} = -1$ $\quad C_{31} = (-1)^4 \begin{vmatrix} -1 & 1 \\ 2 & -2 \end{vmatrix} = 0$

$C_{12} = (-1)^3 \begin{vmatrix} 1 & -2 \\ 3 & -2 \end{vmatrix} = -4$ $\quad C_{22} = (-1)^4 \begin{vmatrix} 0 & 1 \\ 3 & -2 \end{vmatrix} = -3$ $\quad C_{32} = (-1)^5 \begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = 1$

$C_{13} = (-1)^4 \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5$ $\quad C_{23} = (-1)^5 \begin{vmatrix} 0 & -1 \\ 3 & 1 \end{vmatrix} = -3$ $\quad C_{33} = (-1)^6 \begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix} = 1$

$$C_{kof}^{T} = \begin{bmatrix} -2 & -1 & 0 \\ -4 & -3 & 1 \\ -5 & -3 & 1 \end{bmatrix}, \quad C^{-1} = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 3 & -1 \\ 5 & 3 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 3 & -1 \\ 5 & 3 & -1 \end{bmatrix}$$ rješenje

3. Riješiti matričnu jednačinu $(A+B)^{-1}AX^{-1} = B^{-1}$ gdje su matrice $A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$ i $B = \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}$.

Rj.

$(A+B)^{-1}A\cdot X^{-1} = B^{-1} \quad /\cdot(A+B)$ sa lijeve strane

$(A+B)(A+B)^{-1}\cdot A\cdot X^{-1} = (A+B)\cdot B^{-1}$

$A\cdot X^{-1} = (A+B)\cdot B^{-1} \quad /\cdot A^{-1}$ sa lijeve strane

$A^{-1}\cdot A\cdot X^{-1} = A^{-1}(A+B)\cdot B^{-1}$

$X^{-1} = A^{-1}(A+B)\cdot B^{-1} \quad /^{-1}$

$X = B(A+B)^{-1}A$

$$C = A + B = \begin{bmatrix} 4 & -1 \\ 1 & 3 \end{bmatrix}$$

$C=\begin{bmatrix}4&-1\\1&3\end{bmatrix}$, $C^{-1}=\frac{1}{\det C}C_{kof}^{T}$, $\det C=\begin{vmatrix}4&-1\\1&3\end{vmatrix}=13$,

$C_{11}=(-1)^2\cdot 3=3$
$C_{12}=(-1)^3\cdot 1=-1$
$C_{21}=(-1)^3\cdot(-1)=1$
$C_{22}=(-1)^4\cdot 4=4$

$C_{kof}^{T}=\begin{bmatrix}3&1\\-1&4\end{bmatrix}$, $C^{-1}=\begin{bmatrix}\frac{3}{13}&\frac{1}{13}\\-\frac{1}{13}&\frac{4}{13}\end{bmatrix}$

$$X=B\cdot C^{-1}\cdot A=\begin{bmatrix}3&0\\-1&2\end{bmatrix}\cdot\frac{1}{13}\cdot\begin{bmatrix}3&1\\-1&4\end{bmatrix}\cdot\begin{bmatrix}1&-1\\2&1\end{bmatrix}=\frac{1}{13}\begin{bmatrix}3&0\\-1&2\end{bmatrix}\begin{bmatrix}5&-2\\7&5\end{bmatrix}=\frac{1}{13}\begin{bmatrix}15&-6\\9&12\end{bmatrix}$$

$X=\begin{bmatrix}\frac{15}{13}&-\frac{6}{13}\\\frac{9}{13}&\frac{12}{13}\end{bmatrix}$ rješenje matrične jednačine

4. Riješiti matričnu jednačinu $(A+3I)(X-I)=B$, ako je $A=\begin{bmatrix}-2&5&-2\\2&8&0\\-1&-5&-2\end{bmatrix}$, $B=\begin{bmatrix}-3&21&1\\2&50&-2\\1&-22&0\end{bmatrix}$ i I jedinična matrica.

Rj. $(A+3I)(X-I)=B \quad /\cdot(A+3I)^{-1}$ sa lijeve strane

$C^{-1}=\frac{1}{\det C}C_{kof}^{T}$

$(A+3I)^{-1}(A+3I)(X-I)=(A+3I)^{-1}\cdot B$

$X-I=(A+3I)^{-1}\cdot B$

$X=(A+3I)^{-1}\cdot B+I$

$$\det C=\begin{vmatrix}1&5&-2\\2&11&0\\-1&-5&1\end{vmatrix}\ \underline{\underline{I_R+III_R}}=\begin{vmatrix}0&0&-1\\2&11&0\\-1&-5&1\end{vmatrix}=(-1)\begin{vmatrix}2&11\\-1&-5\end{vmatrix}=-1$$

$$C=A+3I=\begin{bmatrix}-2&5&-2\\2&8&0\\-1&-5&-2\end{bmatrix}+\begin{bmatrix}3&0&0\\0&3&0\\0&0&3\end{bmatrix}=\begin{bmatrix}1&5&-2\\2&11&0\\-1&-5&1\end{bmatrix}$$

$C_{11}=(-1)^2\begin{vmatrix}11&0\\-5&1\end{vmatrix}=11$

$C_{12}=(-1)^3\begin{vmatrix}2&0\\-1&1\end{vmatrix}=-2$

$C_{13}=(-1)^4\begin{vmatrix}2&11\\-1&-5\end{vmatrix}=1$

$C_{21}=(-1)^3\begin{vmatrix}5&-2\\-5&1\end{vmatrix}=5$

$C_{31}=(-1)^4\begin{vmatrix}5&-2\\11&0\end{vmatrix}=22$

$C_{22}=(-1)^4\begin{vmatrix}1&-2\\-1&1\end{vmatrix}=-1$

$C_{32}=(-1)^5\begin{vmatrix}1&-2\\2&0\end{vmatrix}=-4$

$C_{23}=(-1)^5\begin{vmatrix}1&5\\-1&-5\end{vmatrix}=0$

$C_{33}=(-1)^6\begin{vmatrix}1&5\\2&11\end{vmatrix}=1$

$C_{kof}^{T}=\begin{bmatrix}11&5&22\\-2&-1&-4\\1&0&1\end{bmatrix}$

$C^{-1}=\begin{bmatrix}-11&-5&-22\\2&1&4\\-1&0&-1\end{bmatrix}$

$$C^{-1}\cdot B=\begin{bmatrix}-11&-5&-22\\2&1&4\\-1&0&-1\end{bmatrix}\cdot\begin{bmatrix}-3&21&1\\2&50&-2\\1&-22&0\end{bmatrix}=\begin{bmatrix}1&3&-1\\0&4&0\\2&1&-1\end{bmatrix}$$

$$X=(A+3I)^{-1}\cdot B+I=\begin{bmatrix}1&3&-1\\0&4&0\\2&1&-1\end{bmatrix}+\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}2&3&-1\\0&5&0\\2&1&0\end{bmatrix}$$ rješenje matrične jednačine

5. Riješiti matričnu jednačinu $(X^{-1}+B^{-1})^{-1}=AX$ ako su

$$A=\begin{bmatrix}3&3&2\\-4&1&-4\\-3&1&-3\end{bmatrix} \quad i \quad B=\begin{bmatrix}1&0&2\\1&-2&0\\0&-1&1\end{bmatrix}.$$

Rj. $(X^{-1}+B^{-1})^{-1}=AX \quad /\cdot(X^{-1}+B^{-1})$ sa desne strane

$$(X^{-1}+B^{-1})^{-1}\cdot(X^{-1}+B^{-1})=AX(X^{-1}+B^{-1})$$

$$I=A+AXB^{-1}$$

$$AXB^{-1}=I-A \quad /\cdot A^{-1} \text{ sa lijeve str.}, \ \cdot B \text{ sa desne str.}$$

$$A^{-1}\cdot A\cdot X\cdot B^{-1}\cdot B=A^{-1}(I-A)\cdot B$$

$$X=A^{-1}(I-A)\cdot B$$

$$A^{-1}=\frac{1}{\det A}\cdot A_{kof}^{-1}$$

$$\det A=\begin{vmatrix}3&3&2\\-4&1&-4\\-3&1&-3\end{vmatrix}\ \underline{\underline{I_k-III_k}}=\begin{vmatrix}1&3&2\\0&1&-4\\0&1&-3\end{vmatrix}=\begin{vmatrix}1&-4\\1&-3\end{vmatrix}=-3+4=1$$

$$A_{11}=(-1)^2\begin{vmatrix}1&-4\\1&-3\end{vmatrix}=1$$

$$A_{12}=(-1)^3\begin{vmatrix}-4&-4\\-3&-3\end{vmatrix}=0$$

$$A_{13}=(-1)^4\begin{vmatrix}-4&1\\-3&1\end{vmatrix}=-1$$

$$A_{21}=(-1)^3\begin{vmatrix}3&2\\1&-3\end{vmatrix}=11 \qquad A_{31}=(-1)^4\begin{vmatrix}3&2\\1&-4\end{vmatrix}=-14$$

$$A_{22}=(-1)^4\begin{vmatrix}3&2\\-3&-3\end{vmatrix}=-3 \qquad A_{32}=(-1)^5\begin{vmatrix}3&2\\-4&-4\end{vmatrix}=4$$

$$A_{23}=(-1)^5\begin{vmatrix}3&3\\-3&1\end{vmatrix}=-12 \qquad A_{33}=(-1)^6\begin{vmatrix}3&3\\-4&1\end{vmatrix}=15$$

$$A_{kof}=\begin{bmatrix}1&0&-1\\11&-3&-12\\-14&4&15\end{bmatrix}$$

$$A^{-1}=\begin{bmatrix}1&11&-14\\0&-3&4\\-1&-12&15\end{bmatrix},\quad X=A^{-1}(I-A)\cdot B=\begin{bmatrix}1&11&-14\\0&-3&4\\-1&-12&15\end{bmatrix}\begin{bmatrix}-2&-3&-2\\4&0&4\\3&-1&4\end{bmatrix}\begin{bmatrix}1&0&2\\1&-2&0\\0&-1&1\end{bmatrix}$$

$$=\begin{bmatrix}1&11&-14\\0&-3&4\\-1&-12&15\end{bmatrix}\begin{bmatrix}-5&8&-6\\4&-4&12\\2&-2&10\end{bmatrix}=\begin{bmatrix}11&-8&-14\\-4&4&4\\-13&10&12\end{bmatrix}$$ rješenje matrične jednačine

6. Riješiti matričnu jednačinu:

$$X\cdot\begin{bmatrix}2&3&4\\1&2&3\\4&3&2\end{bmatrix}+\begin{bmatrix}1&4&5\\0&1&4\\3&4&2\end{bmatrix}=X\cdot\begin{bmatrix}1&4&5\\0&1&4\\3&4&2\end{bmatrix}.$$

Ako označimo $A=\begin{bmatrix}2&3&4\\1&2&3\\4&3&2\end{bmatrix}$ i $B=\begin{bmatrix}1&4&5\\0&1&4\\3&4&2\end{bmatrix}$ imamo

$$XA+B=XB$$

$$XA-XB=-B$$

$$X(A-B)=-B \quad /\cdot(A-B)^{-1} \text{ sa desne strane}$$

$$X=-B(A-B)^{-1}$$

$$C=A-B=\begin{bmatrix}1&-1&-1\\1&1&-1\\1&-1&0\end{bmatrix}$$

$$\det C=\begin{vmatrix}1&-1&-1\\1&1&-1\\1&-1&0\end{vmatrix}\overset{I_k+II_k}{=\!=}\begin{vmatrix}0&-1&-1\\2&1&-1\\0&-1&0\end{vmatrix}=(-2)\begin{vmatrix}-1&-1\\-1&0\end{vmatrix}=(-2)\cdot(-1)=2$$

$$C_{11}=(-1)^2\begin{vmatrix}1&-1\\-1&0\end{vmatrix}=-1 \qquad C_{21}=1 \qquad C_{31}=2$$

$$C_{12}=(-1)^3\cdot 1=-1 \qquad C_{22}=1 \qquad C_{32}=0$$

$$C_{13}=-2 \qquad C_{23}=0 \qquad C_{33}=2$$

$$C^{-1}=\frac{1}{\det C}\cdot C_{kof}^{T}$$

$$C^{-1}=\frac{1}{2}\begin{bmatrix}-1&1&2\\-1&1&0\\-2&0&2\end{bmatrix}, \qquad X=-B\cdot C^{-1}=-\begin{bmatrix}1&4&5\\0&1&4\\3&4&2\end{bmatrix}\cdot\frac{1}{2}\begin{bmatrix}-1&1&2\\-1&1&0\\-2&0&2\end{bmatrix}$$

$$X=-\frac{1}{2}\begin{bmatrix}-15&5&12\\-9&1&8\\-11&7&10\end{bmatrix}=\begin{bmatrix}\frac{15}{2}&-\frac{5}{2}&-6\\\frac{9}{2}&-\frac{1}{2}&-4\\\frac{11}{2}&-\frac{7}{2}&-5\end{bmatrix}$$ rješenje matrične jednačine

7. Riješiti matričnu jednačinu $(A+X)(B-2I)=A$, ako su $A=\begin{bmatrix}-2&-3&-4\\1&-2&3\\4&3&-2\end{bmatrix}$, $B=\begin{bmatrix}3&-2&1\\1&-1&0\\1&-2&2\end{bmatrix}$, I jedinična matrica.

8. Riješiti matričnu jednačinu $A^{-1}X+B=AX$, ako su $A=\begin{bmatrix}-2&5\\-1&3\end{bmatrix}$; $B=\begin{bmatrix}4&5\\6&7\end{bmatrix}$.

9. Riješiti matričnu jednačinu $(XB^{-1})^{-1}=X^{-1}+A$, ako su $A=\begin{bmatrix}-1&3&1\\1&2&3\\0&3&2\end{bmatrix}$; $B=\begin{bmatrix}1&1&1\\5&5&4\\3&2&1\end{bmatrix}$.

Rješenja:

7. $X=\begin{bmatrix}-2&10&-1\\2&2&-5\\-6&-14&19\end{bmatrix}$

8. $X=\begin{bmatrix}4&5\\6&7\end{bmatrix}$

9. $X=\begin{bmatrix}3&-\frac{3}{2}&-\frac{17}{2}\\1&-1&-5\\0&\frac{5}{2}&\frac{15}{2}\end{bmatrix}$

Data je matrična jednačina $A(X-B)^{-1}=B^{-1}A$ i matrice

$$A=\begin{bmatrix}1&1&1\\0&1&1\\0&-1&1\end{bmatrix} \quad i \quad B=\begin{bmatrix}1&1&0\\2&1&0\\3&-1&2\end{bmatrix}.$$

a) Koji uslov moraju zadovoljavat matrice A i B da bi data jednačina imala rješenje $X=2B$?

b) Riješiti datu jednačinu ako matrice A i B ne zadovoljavaju uslov dobijen pod a)

Rj. a) $A(X-B)^{-1}=B^{-1}A$

$X=2B$

$A\cdot B^{-1}=B^{-1}A$ uslov koji moraju zadovoljavati matrice A i B da bi data matrica imala rješenje $X=2B$.

Uslov možemo pisati i na drugi način:

$A=B^{-1}AB$

ili

$B=A^{-1}\cdot B\cdot A$

b)

$A(X-B)^{-1}=B^{-1}A \quad /\cdot(X-B)$ sa desne str

$B^{-1}A(X-B)=A \quad /\cdot B$ sa lijeve str.

$A(X-B)=BA \quad /\cdot A^{-1}$ sa lijeve str.

$X-B=A^{-1}BA$

$X=A^{-1}BA+B$

i odavde možemo pročitati uslov koji smo dobili pod a)
(ako je $B=A^{-1}BA$ tada jednačina ima rješenje $X=2B$)

Provjerimo da li je $B=A^{-1}BA$.

Nađimo prvo A^{-1}

$$A^{-1}=\frac{1}{\det A}\cdot A_{kof}^{T}$$

$$\det A=\begin{vmatrix}1&1&1\\0&1&1\\0&-1&1\end{vmatrix}=\begin{vmatrix}1&1\\-1&1\end{vmatrix}=2$$

$A_{11}=(-1)^2\begin{vmatrix}1&1\\-1&1\end{vmatrix}=2$ $\quad A_{21}=(-1)^3\begin{vmatrix}1&1\\-1&1\end{vmatrix}=-2$ $\quad A_{31}=(-1)^4\begin{vmatrix}1&1\\1&1\end{vmatrix}=0$

$A_{12}=(-1)^3\begin{vmatrix}0&1\\0&1\end{vmatrix}=0$ $\quad A_{22}=(-1)^4\begin{vmatrix}1&1\\0&1\end{vmatrix}=1$ $\quad A_{32}=(-1)^5\begin{vmatrix}1&1\\0&1\end{vmatrix}=-1$

$A_{13}=(-1)^4\begin{vmatrix}0&1\\0&-1\end{vmatrix}=0$ $\quad A_{23}=(-1)^5\begin{vmatrix}1&1\\0&-1\end{vmatrix}=1$ $\quad A_{33}=(-1)^6\begin{vmatrix}1&1\\0&1\end{vmatrix}=1$

$$A_{kof}=\begin{bmatrix}2&0&0\\-2&1&1\\0&-1&1\end{bmatrix} \qquad A_{kof}^{T}=\begin{bmatrix}2&-2&0\\0&1&-1\\0&1&1\end{bmatrix}$$

$$A^{-1}=2\begin{bmatrix}2&-2&0\\0&1&-1\\0&1&1\end{bmatrix}, \qquad B\cdot A=\begin{bmatrix}1&1&0\\2&1&0\\3&-1&2\end{bmatrix}\begin{bmatrix}1&1&1\\0&1&1\\0&-1&1\end{bmatrix}=\begin{bmatrix}1&2&2\\2&3&3\\3&0&4\end{bmatrix}$$

$$A^{-1}\cdot B\cdot A=2\begin{bmatrix}2&-2&0\\0&1&-1\\0&1&1\end{bmatrix}\begin{bmatrix}1&2&2\\2&3&3\\3&0&4\end{bmatrix}=2\begin{bmatrix}-2&-2&-2\\-1&3&-1\\5&3&7\end{bmatrix}=\begin{bmatrix}-1&-1&-1\\-\frac{1}{2}&\frac{3}{2}&-\frac{1}{2}\\\frac{5}{2}&\frac{3}{2}&\frac{7}{2}\end{bmatrix}$$

ovdje vidimo da matrice A i B ne zadovoljavaju uslov dobijen pod a)

$$X=A^{-1}BA+B=\begin{bmatrix}-1&-1&-1\\-\frac{1}{2}&3/2&-1/2\\5/2&3/2&7/2\end{bmatrix}+\begin{bmatrix}1&1&0\\2&1&0\\3&-1&2\end{bmatrix}=\begin{bmatrix}0&0&-1\\3/2&5/2&-1/2\\11/2&1/2&11/2\end{bmatrix}$$

rješenje matrične jednačine

(#) Riješiti matričnu jednačinu $X \cdot A^{-1} = B^{-1}$ ako su

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 1 & 1 & -4 \end{bmatrix} \quad i \quad B = \begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix}.$$

Rj.

$X \cdot A^{-1} = B^{-1}$ $/ \cdot A$ sa desne strane

$X \underbrace{A^{-1} \cdot A}_{I} = B^{-1} \cdot A$

$X = B^{-1} \cdot A$

$$\det B = \begin{vmatrix} 2 & 1 & -1 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{vmatrix} \underline{\underline{I_R - II_R}}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} = 0 + 1$$

$\det B = 1$

$$B^{-1} = \frac{1}{\det B} B_{kof}^{T}$$

$B_{11} = (-1)^2 \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = 1$ $\quad B_{21} = (-1)^3 \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = -1$ $\quad B_{31} = (-1)^4 \begin{vmatrix} 1 & -1 \\ 1 & -2 \end{vmatrix} = -1$

$B_{12} = (-1)^3 \begin{vmatrix} 2 & -2 \\ -1 & 1 \end{vmatrix} = 0$ $\quad B_{22} = (-1)^4 \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} = 1$ $\quad B_{32} = (-1)^5 \begin{vmatrix} 2 & -1 \\ 2 & -2 \end{vmatrix} = 2$

$B_{13} = (-1)^4 \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} = 1$ $\quad B_{23} = (-1)^5 \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} = -1$ $\quad B_{33} = (-1)^6 \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} = 0$

$$B_{kof} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ -1 & 2 & 0 \end{bmatrix}, \quad B_{kof}^{T} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

$$X = B^{-1} \cdot A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 1 & 1 & -4 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 7 \\ 5 & 4 & -7 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\begin{matrix} 2-3-1 & 0+3+2 & 2-3+0 \\ 3-2-1 & 0+2+2 & 3-2+0 \\ 4-1+4 & 0+1-8 & 4-1+0 \end{matrix}$$

$$X = \begin{bmatrix} -2 & 0 & 7 \\ 5 & 4 & -7 \\ -1 & 1 & 3 \end{bmatrix}$$

traženo rješenje

(#) Riješiti matričnu jednačinu $(A+I)^{-1}\cdot X\cdot(3A+I)=2A$ gdje je I jedinična matrica drugog reda a

$$A=\begin{bmatrix}7 & 8\\ -6 & -7\end{bmatrix}.$$

Rj. $(A+I)^{-1}\cdot X\cdot(3A+I)=2A \quad /(A+I)$ sa lijeve strane

$X\cdot(3A+I)=(A+I)\cdot 2A \quad /\cdot(3A+I)^{-1}$ sa desne strane

$X=(A+I)\cdot 2A\cdot(3A+I)^{-1}$

$$A=\begin{bmatrix}7 & 8\\ -6 & -7\end{bmatrix} \qquad I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \qquad A+I=\begin{bmatrix}8 & 8\\ -6 & -6\end{bmatrix}$$

$20\cdot 22$
40
40
440

$$3A+I=\begin{bmatrix}22 & 24\\ -18 & -20\end{bmatrix} \qquad 3A=\begin{bmatrix}21 & 24\\ -18 & -21\end{bmatrix}$$

$18\cdot 24$
72
36
432

Označimo sa $B=3A-I$ pa pronađimo B^{-1}

$$B^{-1}=\frac{1}{\det B}\cdot B_{kof}^{T} \qquad \det B=\begin{vmatrix}22 & 24\\ -18 & -20\end{vmatrix}=-440+432=-8$$

$B_{11}=(-1)^2\cdot(-20)=-20$

$B_{12}=(-1)^3\cdot(-18)=18$

$B_{21}=(-1)^3\cdot 24=-24$

$B_{22}=(-1)^4\cdot 22=22$

$$B_{kof}=\begin{bmatrix}-20 & 18\\ -24 & 22\end{bmatrix}$$

$$B^{-1}=\frac{-1}{8}\begin{bmatrix}-20 & -24\\ 18 & 22\end{bmatrix}=(3A+I)^{-1}$$

$$X=(A+I)\cdot 2A\cdot(3A+I)^{-1}=\begin{bmatrix}8 & 8\\ -6 & -6\end{bmatrix}\cdot 2\cdot\begin{bmatrix}7 & 8\\ -6 & -7\end{bmatrix}\cdot\frac{-1}{8}\begin{bmatrix}-20 & -24\\ 18 & 22\end{bmatrix}$$

$$=2\begin{bmatrix}4 & 4\\ -3 & -3\end{bmatrix}\cdot 2\begin{bmatrix}7 & 8\\ -6 & -7\end{bmatrix}\cdot\frac{-1}{8}\cdot 2\begin{bmatrix}-10 & -12\\ 9 & 11\end{bmatrix}=8\cdot\frac{-1}{8}\begin{bmatrix}4 & 4\\ -3 & -3\end{bmatrix}\begin{bmatrix}7 & 8\\ -6 & -7\end{bmatrix}\begin{bmatrix}-10 & -12\\ 9 & 11\end{bmatrix}$$

$$=(-1)\begin{bmatrix}4 & 4\\ -3 & -3\end{bmatrix}\begin{bmatrix}-10 & -12\\ 9 & 11\end{bmatrix}=(-1)\begin{bmatrix}-4 & -4\\ 3 & 3\end{bmatrix}=\begin{bmatrix}4 & 4\\ -3 & -3\end{bmatrix}$$ rješenje matrične jednačine

(#) Riješiti matričnu jednačinu $(AXB)^{-1} = B^{-1}(X^{-1}+B)$ ako je

$$A = \begin{bmatrix} 3 & -4 & 5 \\ 2 & -3 & 1 \\ 3 & -5 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}.$$

Rj.

$$(AXB)^{-1} = B^{-1}(X^{-1}+B)$$

$$B^{-1}X^{-1}A^{-1} = B^{-1}X^{-1} + B^{-1}\cdot B \quad / \cdot B \text{ sa lijeve strane}$$

$$X^{-1}A^{-1} = X^{-1} + B$$

$$X^{-1}A^{-1} - X^{-1} = B$$

$$X^{-1}(A^{-1} - I) = B \quad / \cdot (A^{-1}-I)^{-1} \text{ sa desne strane}$$

$$X^{-1} = B(A^{-1}-I)^{-1} \quad /^{-1}$$

$$X = (A^{-1}-I)\cdot B^{-1}$$

$$\det A = \begin{vmatrix} 3 & -4 & 5 \\ 2 & -3 & 1 \\ 3 & -5 & -1 \end{vmatrix} \overset{I_k + II_k}{=} \begin{vmatrix} -1 & -4 & 5 \\ -1 & -3 & 1 \\ -2 & -5 & -1 \end{vmatrix} \overset{II_v - I_v}{\underset{III_v - I_v\cdot 2}{=}} \begin{vmatrix} -1 & -4 & 5 \\ 0 & 1 & -4 \\ 0 & 3 & -11 \end{vmatrix} = (-1)\begin{vmatrix} 1 & -4 \\ 3 & -11 \end{vmatrix} = (-1)(-11+12) = -1$$

$$A^{-1} = \frac{1}{\det A}\cdot A_{kof}^{T}$$

$A_{11} = (-1)^2 \begin{vmatrix} -3 & 1 \\ -5 & -1 \end{vmatrix} = 3+5 = 8$ $\quad A_{21} = (-1)^3 \begin{vmatrix} -4 & 5 \\ -5 & -1 \end{vmatrix} = -(4+25) = -29 \quad A_{31} = 11$

$A_{12} = (-1)^3 \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} = -(-2-3) = 5$ $\quad A_{22} = (-1)^4 \begin{vmatrix} 3 & 5 \\ 3 & -1 \end{vmatrix} = -3-15 = -18 \quad A_{32} = 7$

$A_{13} = (-1)^4 \begin{vmatrix} 2 & -3 \\ 3 & -5 \end{vmatrix} = -10+9 = -1$ $\quad A_{23} = (-1)^5 \begin{vmatrix} 3 & -4 \\ 3 & -5 \end{vmatrix} = -(-15+12) = 3 \quad A_{33} = -1$

$$A_{kof} = \begin{bmatrix} 8 & 5 & -1 \\ -29 & -18 & 3 \\ 11 & 7 & -1 \end{bmatrix}. \quad A^{-1} = (-1)\begin{bmatrix} 8 & -29 & 11 \\ 5 & -18 & 7 \\ -1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} -8 & 29 & -11 \\ -5 & 18 & -7 \\ 1 & -3 & 1 \end{bmatrix}.$$

$$\det B = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{vmatrix} \overset{II_v - I_v\cdot 2}{\underset{III_v - I_v\cdot 2}{=}} \begin{vmatrix} 1 & 2 & 2 \\ 0 & -3 & -6 \\ 0 & -6 & -3 \end{vmatrix} = \begin{vmatrix} -3 & -6 \\ -6 & -3 \end{vmatrix} = 9-36 = -27$$

$$B^{-1} = \frac{1}{\det B}\cdot B_{kof}^{T} \underset{\text{ZA VJEŽBU (slično)}}{=} \frac{(-1)}{-27}\begin{bmatrix} 3 & 6 & 6 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix} = \frac{1}{27}\cdot 3\cdot \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$B^{-1} = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$A^{-1} - I = \begin{bmatrix} -8 & 29 & -11 \\ -5 & 18 & -7 \\ 1 & -3 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -9 & 29 & -11 \\ -5 & 17 & -7 \\ 1 & -3 & 0 \end{bmatrix}$$

$$X = (A^{-1} - I) \cdot B^{-1} = \begin{bmatrix} -9 & 29 & -11 \\ -5 & 17 & -7 \\ 1 & -3 & 0 \end{bmatrix} \cdot \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} =$$

$$= \frac{1}{9}\begin{bmatrix} 27 & 33 & -87 \\ 15 & 21 & -51 \\ -5 & -1 & 8 \end{bmatrix}$$

$$X = \begin{bmatrix} 3 & 11/3 & -\frac{29}{3} \\ 5/3 & 7/3 & -17/3 \\ -\frac{5}{9} & -\frac{1}{9} & \frac{8}{9} \end{bmatrix}$$ rješenje matrične jednačine

Riješiti matričnu jednačinu $A \cdot X^{-1} \cdot B = B \cdot A$, ako je $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ i $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

Rj.

$A X^{-1} B = B \cdot A \quad / \cdot A^{-1}$ sa lijeve strane

$X^{-1} B = A^{-1} B \cdot A \quad / \cdot B^{-1}$ sa desne strane

$X^{-1} = A^{-1} B \cdot A \cdot B^{-1} \quad /^{-1}$

$X = B A^{-1} B^{-1} A$

$$A^{-1} = \frac{1}{\det A} \cdot A_{kof}^T \qquad \det A = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \qquad A_{kof} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$A_{11} = 1 \quad A_{21} = -1$

$A_{12} = 0 \quad A_{22} = 1$

$$A_{kof}^T = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$B^{-1} = \frac{1}{\det B} \cdot B_{kof}^T$$

$B_{11} = 1$

$B_{12} = -1$

$B_{21} = 0$

$B_{22} = 1$

$$B_{kof} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \qquad B_{kof}^T = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$\det B = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

$$B \cdot A^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$B^{-1} \cdot A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$X = BA^{-1} \cdot B^{-1} A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$ traženo rješenje

(#) Riješiti matričnu jednačinu: $AX - 2B = 3X + A$ gdje je

$$A = \begin{bmatrix} 6 & 1 & 0 \\ 0 & 5 & 2 \\ 0 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 0 \\ 2 & 3 & 1 \\ 4 & 0 & 3 \end{bmatrix}.$$

Rj.

$$AX - 2B = 3X + A$$

$$AX - 3X = 2B + A$$

$$\underbrace{(A - 3I)}_{M} X = \underbrace{2B + A}_{N}$$

$$MX = N \quad / \cdot M^{-1} \text{ sa lijeve str.}$$

$$M^{-1} M X = M^{-1} \cdot N$$

$$X = M^{-1} \cdot N$$

$$M = A - 3I = \begin{bmatrix} 6 & 1 & 0 \\ 0 & 5 & 2 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$N = 2B + A = \begin{bmatrix} -2 & 4 & 0 \\ 4 & 6 & 2 \\ 8 & 0 & 6 \end{bmatrix} + \begin{bmatrix} 6 & 1 & 0 \\ 0 & 5 & 2 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 0 \\ 4 & 11 & 4 \\ 8 & 0 & 10 \end{bmatrix}$$

$$M^{-1} = \frac{1}{\det M} M_{kof}^{T}$$

$$\det M = \begin{vmatrix} 3 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 3 \cdot 2 \cdot 1 = 6$$

$$M_{11} = (-1)^2 \begin{vmatrix} 2 & 2 \\ 0 & 1 \end{vmatrix} = 2 \qquad M_{21} = (-1)^3 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -1$$

$$M_{12} = (-1)^3 \begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix} = 0 \qquad M_{22} = (-1)^4 \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} = 3$$

$$M_{13} = (-1)^4 \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = 0 \qquad M_{23} = (1)^5 \begin{vmatrix} 3 & 1 \\ 0 & 0 \end{vmatrix} = 0$$

$$M_{31} = (-1)^4 \begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix} = 2$$

$$M_{32} = (-1)^5 \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = -6$$

$$M_{33} = (-1)^6 \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} = 6$$

$$M_{kof} = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 0 \\ 2 & -6 & 6 \end{bmatrix}, \quad M_{kof}^{T} = \begin{bmatrix} 2 & -1 & 2 \\ 0 & 3 & -6 \\ 0 & 0 & 6 \end{bmatrix}$$

$$M^{-1} = \frac{1}{6} \begin{bmatrix} 2 & -1 & 2 \\ 0 & 3 & -6 \\ 0 & 0 & 6 \end{bmatrix},$$

$$X = M^{-1} \cdot N = \frac{1}{6} \begin{bmatrix} 2 & -1 & 2 \\ 0 & 3 & -6 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 4 & 5 & 0 \\ 4 & 11 & 4 \\ 8 & 0 & 10 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 20 & -1 & 16 \\ -36 & 33 & 48 \\ 48 & 0 & 60 \end{bmatrix}$$

$8 - 4 + 16$
$10 - 11 + 0$
$0 - 4 + 20$

$0 + 12 - 48$
$0 + 33 + 0$
$12 - 60$

$$X = \begin{bmatrix} \frac{10}{3} & -\frac{1}{6} & \frac{8}{3} \\ -6 & \frac{11}{2} & 8 \\ 8 & 0 & 10 \end{bmatrix}$$ traženo rješenje

\# Riješiti matričnu jednačinu $(XA+B)^{-1}(XC+B)=C$, ako je $A=\begin{bmatrix}-1 & 2 & 3\\ 0 & 2 & -1\\ 0 & 0 & 3\end{bmatrix}$, $B=\begin{bmatrix}2 & -1 & 1\\ 0 & -1 & 1\\ 0 & 0 & 1\end{bmatrix}$ i $C=\begin{bmatrix}-1 & 1 & 2\\ 0 & -1 & 2\\ 0 & 0 & 2\end{bmatrix}$

Rj. $(XA+B)^{-1}(XC+B)=C \quad /(XA+B)$ sa lijeve strane

$$\underbrace{(XA+B)(XA+B)^{-1}}_{I}(XC+B)=(XA+B)\cdot C$$

$$XC+B=XAC+BC$$

$$XC-XAC=BC-B$$

$$X(C-AC)=BC-B \quad /(C-AC)^{-1} \text{ sa desne strane}$$

$$X=B(C-I)(C-AC)^{-1}$$

$$C-I=\begin{bmatrix}-2 & 1 & 2\\ 0 & -2 & 2\\ 0 & 0 & 1\end{bmatrix}$$

$$B(C-I)=\begin{bmatrix}2 & -1 & 1\\ 0 & -1 & 1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-2 & 1 & 2\\ 0 & -2 & 2\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}-4 & 4 & 3\\ 0 & 2 & -1\\ 0 & 0 & 1\end{bmatrix}$$

$$AC=\begin{bmatrix}-1 & 2 & 3\\ 0 & 2 & -1\\ 0 & 0 & 3\end{bmatrix}\begin{bmatrix}-1 & 1 & 2\\ 0 & -1 & 2\\ 0 & 0 & 2\end{bmatrix}=\begin{bmatrix}1 & -3 & 8\\ 0 & -2 & 2\\ 0 & 0 & 6\end{bmatrix}$$

Označimo sa $D=C-AC=\begin{bmatrix}-2 & 4 & -6\\ 0 & 1 & 0\\ 0 & 0 & -4\end{bmatrix}$

Izračunajmo D^{-1}: $\quad D^{-1}=\dfrac{1}{\det D}D_{kof}^{T}$

$D_{11}=(-1)^2\begin{vmatrix}1 & 0\\ 0 & -4\end{vmatrix}=-4$ $\quad D_{21}=(-1)^3\begin{vmatrix}4 & -6\\ 0 & -4\end{vmatrix}=16$ $\quad D_{31}=(-1)^4\begin{vmatrix}4 & -6\\ 1 & 0\end{vmatrix}=6$

$D_{12}=(-1)^3\begin{vmatrix}0 & 0\\ 0 & -4\end{vmatrix}=0$ $\quad D_{22}=(-1)^4\begin{vmatrix}-2 & -6\\ 0 & -4\end{vmatrix}=8$ $\quad D_{32}=(-1)^5\begin{vmatrix}-2 & -6\\ 6 & 0\end{vmatrix}=0$

$D_{13}=(-1)^4\begin{vmatrix}0 & 1\\ 0 & 0\end{vmatrix}=0$ $\quad D_{23}=(-1)^5\begin{vmatrix}-2 & 4\\ 0 & 0\end{vmatrix}=0$ $\quad D_{33}=(-1)^6\begin{vmatrix}-2 & 4\\ 0 & 1\end{vmatrix}=-2$

$$\det D=\begin{vmatrix}-2 & 4 & -6\\ 0 & 1 & 0\\ 0 & 0 & -4\end{vmatrix}=(-4)\begin{vmatrix}-2 & 4\\ 0 & 1\end{vmatrix}=8 \qquad D_{kof}=\begin{bmatrix}-4 & 0 & 0\\ 16 & 8 & 0\\ 6 & 0 & -2\end{bmatrix} \qquad D_{kof}^{T}=\begin{bmatrix}-4 & 16 & 6\\ 0 & 8 & 0\\ 0 & 0 & -2\end{bmatrix}$$

$$D^{-1}=\begin{bmatrix}-\frac{1}{2} & 2 & \frac{3}{4}\\ 0 & 1 & 0\\ 0 & 0 & -\frac{1}{4}\end{bmatrix}, \quad X=B(C-I)(C-AC)^{-1}=\frac{1}{8}\begin{bmatrix}-4 & 4 & 3\\ 0 & 2 & -1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-4 & 16 & 6\\ 0 & 8 & 0\\ 0 & 0 & -2\end{bmatrix}$$

$$X=\frac{1}{8}\begin{bmatrix}16 & -32 & -30\\ 0 & 16 & 2\\ 0 & 0 & -2\end{bmatrix}=\begin{bmatrix}2 & -4 & -\frac{15}{4}\\ 0 & 2 & \frac{1}{4}\\ 0 & 0 & -\frac{1}{4}\end{bmatrix} \text{ traženo rješenje}$$

(#) Riješiti matričnu jednačinu $XAB=C$, $A=\begin{bmatrix}1&0&1\\2&1&1\\0&1&1\end{bmatrix}$, $B=\begin{bmatrix}1&1&-1\\0&1&2\\-1&1&1\end{bmatrix}$, $C=\begin{bmatrix}0&4&4\end{bmatrix}$.

Rj.

$XAB=C \quad /\cdot(AB)^{-1}$ sa desne strane

$X(AB)(AB)^{-1}=C\cdot(AB)^{-1}$

$X=C\cdot(AB)^{-1}$

$$AB=\begin{bmatrix}1&0&1\\2&1&1\\0&1&1\end{bmatrix}\begin{bmatrix}1&1&-1\\0&1&2\\-1&1&1\end{bmatrix}=\begin{bmatrix}0&2&0\\1&4&1\\-1&2&3\end{bmatrix}$$

$$\det(AB)=\begin{vmatrix}0&2&0\\1&4&1\\-1&2&3\end{vmatrix}=(-2)\begin{vmatrix}1&1\\-1&3\end{vmatrix}=(-2)(3+1)=-8$$

AB označimo sa M, nađimo M^{-1}

$M_{11}=(-1)^2\begin{vmatrix}4&1\\2&3\end{vmatrix}=10$ $\quad M_{21}=(-1)^3\begin{vmatrix}2&0\\2&3\end{vmatrix}=-6$ $\quad M_{31}=(-1)^4\begin{vmatrix}2&0\\4&1\end{vmatrix}=2$

$M_{12}=(-1)^3\begin{vmatrix}1&1\\-1&3\end{vmatrix}=-4$ $\quad M_{22}=(-1)^4\begin{vmatrix}0&0\\-1&3\end{vmatrix}=0$ $\quad M_{32}=(-1)^5\begin{vmatrix}0&0\\1&1\end{vmatrix}=0$

$M_{13}=(-1)^4\begin{vmatrix}1&4\\-1&2\end{vmatrix}=6$ $\quad M_{23}=(-1)^5\begin{vmatrix}0&2\\-1&2\end{vmatrix}=-2$ $\quad M_{33}=(-1)^6\begin{vmatrix}0&2\\1&4\end{vmatrix}=-2$

$$M_{kof}=\begin{bmatrix}10&-4&6\\-6&0&-2\\2&0&-2\end{bmatrix},\quad M_{kof}^T=\begin{bmatrix}10&-6&2\\-4&0&0\\6&-2&-2\end{bmatrix}$$

$$(AB)^{-1}=\frac{1}{\det M}\cdot M_{kof}^T=\frac{-1}{8}\begin{bmatrix}10&-6&2\\-4&0&0\\6&-2&-2\end{bmatrix}=\begin{bmatrix}-5/4&3/4&-1/4\\1/2&0&0\\-3/4&1/4&1/4\end{bmatrix}$$

$$X=C\cdot(AB^{-1})=\begin{bmatrix}0&4&4\end{bmatrix}\cdot\left(-\frac{1}{8}\right)\begin{bmatrix}10&-6&2\\-4&0&0\\6&-2&-2\end{bmatrix}=\left(-\frac{1}{8}\right)\begin{bmatrix}8&-8&-8\end{bmatrix}$$

$X=\begin{bmatrix}-1&1&1\end{bmatrix}$ rješenje matrične jednačine